OCT 23 1925

Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS
IN JUNIOR AND SENIOR HIGH SCHOOLS

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Published by the
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
NEW YORK YONKERS, N. Y.

Entered as second-class matter, November 10, 1921, at the Post Office at Yonkers, N. Y., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized November 17, 1921.

THE MATHEMATICS TEACHER

VOLUME XVIII OCTOBER, 1925 NUMBER 6

THE SEQUENCE OF THEOREMS IN SCHOOL GEOMETRY*

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The purpose of this article is to restate and develop certain proposals set out in an address to the Mathematical Association of Great Britain given at the annual meeting held in January, 1917.

I assume as common ground that the school course in geometry should show two main divisions: (1) a heuristic stage in which the chief purpose is to order and clarify the spatial experiences which the pupil has gained from his everyday intercourse with the physical world, to explore the more salient and interesting properties of figures, and to illustrate the useful applications of geometry, as in surveying and "Mongean" geometry; (2) a stage in which the chief purpose is to organize into some kind of logical system the knowledge gained in the earlier stage and to develop it further. In the first stage obvious truths (such as the transversal properties of parallel lines) are freely taken for granted, and deduction is employed mainly to derive from them important and striking truths (such as the constancy of the angle-sum of a triangle) which are not forced upon us by observation. The second stage is marked by an attempt, more or less thoroughgoing and "rigorous," to explore the connexions between geometrical truths and to exhibit them as the logical consequences of a few simple principles.

About the first stage I shall say nothing except to urge (i) that its range should be liberal, including the simpler truths of tri-dimensional geometry and the properties of figures, such as the conic sections, which were excluded from the Euclidean canon, and (ii) that it should occupy the pupil until he is mature enough really to profit by the second stage—which I interpret as meaning that he is not more than two years from the University Entrance examination. It is upon the second stage that I wish to concentrate attention.

*Reprinted (with slight changes) from the *Mathematical Gazette* (London) for May, 1922, by kind permission of its Editor.

The central purpose of this stage is, we have said, to develop the logic of geometry. A simple illustration may make that purpose clear. Let *ABCD* be a quadrilateral in which the angles at *A*, *B*, *C*, are right angles by construction; then it may be proved that *D* is also a right angle. What does the word "prove" mean here? It does not mean that argument is needed to make a boy believe the statement, for there is hardly anything more obviously true. What it means is that the rectangularity of *D* can be shown not to be an isolated fact but a logical consequence of truths (the fundamental congruence-theorem and the parallel postulate) which he has already accepted. In other words, the argument is not for conviction, but to bring out the logical structure of this particular region of geometrical truth.* Indeed, the example illustrates not inaptly Russell and Whitehead's dictum that we have often more reason to believe our axioms true because true consequences flow from them than to believe in the consequences because they flow from the axioms.

Now, if a boy is to gain profit from logical geometry he must in the first place have a degree of mental maturity which is rarely reached before adolescence, and in the second place its purpose must be carefully explained to him. Good teachers, no doubt, always do explain it, though the requirement is ignored in most of the text-books. But if the explanation is to be really satisfatcory it must, I submit, be more philosophical than is usually the case. Pray do not jib at the word philosophical. I mean nothing worse than this: that we should take a good deal of pains to make our pupils realize clearly the logical achitecture of the geometrical system. Treated in a sufficiently broad and concrete way, the subject is a fascinating one, appealing strongly to a boy's curiosity and imagination. If he does not get a reasonable amount of intellectual satisfaction out of it, we have a clear indication that he ought not to be doing logical geometry at all.

What is the "logical architecture" of the ordinary geometrical system? It has three main features. The first consists of certain fundamental properties of points, lines and planes—for in-

*I do not, of course, deny that the logical coherence of a geometrical system fortifies our belief in the truth of all its parts; my point is merely that this result of a logical inquiry into geometry is not the main reason why we undertake it. To avoid another possible source of misunderstanding, I add that I deliberately ignore here, as too abstract for the schoolboy, the standpoint of the truly "pure" geometer. The geometry I have in view is the scientific study of actual space.

stance, the fact that a plane is determined by three non-collinear points and that two planes intersect in a straight line. These are the "foundations of geometry," and, as you know, have been the object of an immense amount of patient and subtle scrutiny in recent years. The second feature comprises the axioms and theorems about congruence, especially the congruence of triangles. The third is, in the Euclidean system, some form of the parallel postulate—now-a-days usually the axiom (improperly) called Playfair's. About the last feature I shall shortly make a proposal which is the *fons et origo* of this discourse. But let us proceed towards it in an orderly way.

The study of the foundations of geometry presupposes a logical faculty far more developed than it can generally be in the boy of fourteen or fifteen. At the beginning of the logical stage it must suffice to call attention to the obvious properties of lines and planes and to point out that they are to be assumed in what follows. The usual theorems about the angles between intersecting lines form a natural appendix to the discussion.

In Euclid's *Elements* the theorems about the congruence of triangles occur at intervals in the first book. Modern text-books rightly bring them together and by that means emphasize their importance and their significance for the geometrical argument. But it is still usual to follow Euclid in proving them by the method of superposition, notwithstanding the severity with which modern geometers have criticized it. Mr. Bertrand Russell, who asserts that it "strikes every intelligent child as a juggle," possibly overestimates the intelligence of children, but does not exaggerate its defects as a principle of proof. The following argument indicates the substance of his objections.

Let two houses be built of the same materials and from the same plans, one (say) in New York, the other in London, and suppose the genius of Aladdin's lamp, in sportive mood, to remove the New York house one night and to replace it by the one from London. Then, though the occupants would no doubt receive a severe shock next morning, passersby would be quite unaware that anything had happened. The argument from superposition says we may conclude that because the London house, when it reaches New York, exactly fills the place of the London house, it filled a space of exactly the same size and shape before it left London. However true the conclusion may be, it

certainly does not follow from the premises. The fact that the London house replaces perfectly the one at New York proves at most that the same plans carried out twice over *on the same spot*, produce results geometrically indistinguishable; it cannot prove that the London house, while still in England, was "congruent" with its fellow at New York. Our belief in that congruence has, surely, an earlier and deeper foundation. Suppose that you were in London and in the confidence of the humorous demon. Then, if you knew that the two houses had been built from the same plans, you would be certain, *before the event*, that the London house would exactly replace the one at New York. And, if it proved not to do so, you would doubt, not the basis of your conviction, but the workmanship of the builders.

Considerations of this kind suggest that in the interests of clear thinking we should give up the argument from superposition, and place the logical treatment of congruence on its real foundations. Those, I submit, are: (1) the (assumed) possibility that a figure, occurring anywhere, may be exactly repeated anywhere else, and (2) the (assumed) fact that certain elementary constructions, such as drawing a line through two given points, measuring off a given length along a ray, or setting off an angle of given magnitude, can be carried out in only one way. Fusing these assumptions into one, we have the Principle of Congruence: namely, that figures produced by combining the aforesaid elementary constructions in any given (unambigious) manner are all geometrically equivalent—or, in simpler language, that if a given geometrical construction can be carried out in only one way it produces equivalent figures whether carried out *here* or *there*. For instance, on one side of a given line AB, of length c, it is clearly possible to construct only one triangle ABC, such that AC is of given length b, and the angle between AB and AC of a given magnitude A. It follows from the principle of congruence that triangles drawn to this specification must be equivalent or congruent wherever they may be.

D. Hilbert, in his well-known book,* follows this principle, but limits it to the following axiom: If in two triangles $A'B' = AB$, $A'C' = AC$ and $\angle A' = \angle A$, then $\angle B' = \angle B$ and $\angle C' = \angle C$. From this assumption the equality of BC and

*The *Foundations of Geometry*. A translation is published by the Open Court Publishing Company.

$B'C'$ follows and (eventually) all the other congruence-theorems which do not involve the parallel postulate. † I venture to think Hilbert's limitation too drastic for school use. A boy will gain what I have called a more philosophical view of geometry if he is taught to apply the principle in the broader form. For example, it should be used to prove that triangles are congruent not only when they have two sides in each and the included angles equal, but also when they have their base equal and the angles at the base equal in each. From the former theorem it is easy to deduce the equality of the base angles in an isosceles triangle and the congruence of two triangles in which the sides of one are respectively equal to the sides of the other. With these theorems in our hands, we have all the tools needed to develop geometry as far as the principle of congruence alone can support it.

It is, I submit, of great importance to bring out clearly that though the principle of congruence accounts for many striking properties of figures, it does not account for all. For instance, it does not explain the properties of parallograms, nor Pythagoras' theorem, nor why the angles of a triangle always add up to 180°, nor even the perfectly obvious fact that all the angles of a square must be right angles. For more than two thousand years mathematicians sought to bring these properties within its purview, but were always baffled. Finally, at the beginning of the nineteenth century, it became clear that the problem is insoluble, and that the intractable truths to which I have referred depend upon a second great property of space, additional to and in that sense independent of the property which makes congruent figures possible.

Now what I specially wish to discuss to-night is the question how this second great property should be formulated. Euclid, of course, expressed it in his parallel postulate ("Postulate 5" or "Axiom 12"), and so started a tradition which has been almost universally followed. To challenge a policy laid down by so great a man and hallowed by centuries of acceptance is, I admit, an audacious act. I venture, however, humbly to suggest that Euclid might have served the world better if he had followed another line, and I am about to urge that we should take

†He might equally well have started with the assumption that if $AB = A'B'$, $\angle A = \angle A'$ and $\angle B = \angle B'$, then $AC = A'C'$ and $BC = B'C'$ From this it follows that $\angle C = \angle C'$, and Euclid I. 4 can also be deduced.

that line now. I shall argue the question from the standpoint of one whose main interest is in the *teaching* of geometry, but I believe there is a great deal to be said for the proposal from the purely scientific point of view.

I will begin by stressing again the importance of making boys understand clearly the architecture of geometry. From the teaching standpoint it is a serious weakness of Euclid's procedure that he introduces the parallel postulate only to prove the *converse* of a theorem, and thereafter (I think) mentions it again only once. It is proved (I. 27) that if a transversal cuts two lines at the same angle the lines cannot meet; but, to get on, we must also know that, if they do not meet, any transversal cuts them both at the same angle. It is to guarantee this conclusion (I.29) that Axiom 12 (or Playfair's equivalent) makes its appearance. It needs little psychology to see that this procedure does not give the axiom a fair chance. Everyone knows how slow boys are to be convinced that, although a primary proposition has been proved, the truth of its converse remains an open question. It follows that the entrance of the new axiom into the geometrical scheme is psychologically inconspicuous, and that the momentous consequences of admitting it are not clearly realized.

Now, from the list of truths not deducible from the principle of congruence, we have omitted by far the most important instance: namely, the existence and properties of similar figures. That figures of very different sizes may yet have exactly the same shape is a fact borne in upon a child from his earliest hours. His mother, as she approaches his cradle, is presented (to use the psychological term) as a series of such figures; it needs no argument or persuasion to make him recognize the cat and the dog in his picture-book; and at a later stage he accepts maps and plans as the most natural things in the world.* In short, the main facts about similar figures are so familiar, so interesting and so useful in their applications that most good teachers now give them a conspicuous place in the heuristic stage of geometry.† I cannot see how it can be denied that they are most clearly entitled to it and that it is a serious error to ignore them. If

*The difficulty with regard to maps is, indeed, to persuade him that they are not merely reduced diagrams of the areas they represent.

†The work should include the enlargement and reduction of drawings and a simple treatment of perspective and should incidentally teach the correct technical use of the term "similar."

teachers were wholly free to obey their teaching sense, they would probably give similarity a still more important position in their schemes of work. What chiefly deters them is the unhappy tradition which postponed the *logical* treatment of the subject until so late a point that the majority of boys never reached it. Though the Euclidean proofs are gone, the tradition still operates. Its evil effects will not wholly disappear unless we give to the principle of similarity a place in logical geometry corresponding to its psychological importance and its value as an instrument of investigation. That is what I wish to do. Instead of deducing the existence of similar figures from the (assumed) existence of parallels, I propose that we shall deduce the existence and properties of parallels from the (assumed) existence of similar figures. I make the proposal for the reasons explained in this paragraph; and I shall proceed to defend it by arguing (i) that the change would avoid the weakness pointed out in the preceding paragraph, *i.e.* would make the architecture of elementary geometry much clearer, and (ii) that the proofs needed are at least as easy as the proofs they replace, and are sometimes much simpler.

With regard to the first point. The best way to show that one property of space is independent of another is to invent a kind of space in which the latter exists but the former is absent. That is, in effect, what Lobatchevsky did to prove the independence of the parallel postulate. His argument is far too difficult for ordinary school-boys,§ but it is easy to show that congruence may exist where similarity is absent. Take a sphere of such a size that a degree along its equator measures one inch, and draw on it a triangle composed of great circular arcs measuring respectively 6, 8 and 10 inches. (Great circular arcs are not, of course, straight lines, but they correspond to them inasmuch as they mark out the shortest path between two points on a sphere.) If a triangle with sides of these lengths were drawn on a flat sheet of paper, the angles would be 36° 52′, 53° 8′ and 90°, of which the sum is exactly 180°. On the globe the corresponding angles are 36° 54′, 53° 10′ and 90° 2′, making a total of 180° 6′. Thus the triangle on the sphere is slightly different in

§I concur with Mr. Carson's opinion (*Mathematical Education*, p. 104) that everyone who proceeds to a University "should gain some slight idea of the nature of non-Eulidean geometry," and I submit that what follows here is not a bad introduction to the subject.

form from the triangle on the flat; but, like the latter, it would have exactly the same shape wherever drawn. That is to say, the surface of a sphere resembles a plane in admitting endless repetition of the same figure.

The important difference comes into view when we attempt to reproduce the triangle on a different scale—making the sides, for instance, four times as long. On the flat sheet this can be done without disturbing the shape of the triangle, but on the sphere it is otherwise. For if the sides were lengthened to 24, 32 and 40 inches, the angles would be increased to 39° 13′, 55° 28′ and 92° 19′, making a total of 187°. Thus the surface of a sphere, being uniform, allows of congruent figures yet does not possess the property which makes similar figures possible. It is clear, therefore, that there is no necessary connection between the properties of congruence and of similarity. Space as we know it appears to possess both, but it might conceivably have had the first without also having the second.

In conformity with this conclusion it is then, proposed to organize the whole of geometry on the basis of two (assumed) properties of space. Expressed in popular language they are:

(1) A given figure can be exactly reproduced anywhere.

(2) A given figure can be reproduced anywhere on any (enlarged or diminished) scale.

No further property of space need be assumed, for no property of figures has ever been discovered which cannot be derived from one or both of these.

It cannot, I submit, be denied that logical geometry, built up in the way proposed, would gain greatly in clearness and symmetry. There is no visible kinship between the postulate of congruence and the parallel postulate; but the postulate of similarity resembles and supplements the postulate of congruence in a way that is both obvious and gratifying to the aesthetic sense.

We turn now to the proofs, which are to be based on the axiom that, given a rectilinear figure and any straight line, it is always possible to construct on the given straight line a figure similar to the given figure. Armed with this axiom we can easily show that there are three sets of conditions for similarity between triangles, corresponding, one by one, to the conditions for congruence. The proofs all follow the same lines, so that it will suffice to give the first.

In the triangles ABC, and $A'B'C'$, let $A'B'$ and $A'C'$ have the same ratio to AB and AC respectively, and let $\angle A' = \angle A$. Then we are to prove that $A'B'C'$ is similar to ABC. Now we know by the axiom that there can be drawn upon $A'B'$ a triangle $A'B'C''$ similar to ABC, *having* $\angle A' = \angle A$ and the ratio $A'C'' : AC = A'B' : AB$. It follows that $A'C'' : AC = A'C' : AC$; and hence that the point C'' is identical with C' and this triangle $A'B'C''$ with $A'B'C'$. The triangle $A'B'C'$ must, therefore, be similar to ABC.

You will observe that I have said nothing about the commensurability of the magnitudes. As a matter of fact, I think Legendre was right in maintaining that the measurement of ratios is a question for arithmetic, and that we are not necessarily called upon to discuss it in geometry. If we give the proofs of similarity now current in text-books we *must* say something about it; for those proofs deliberately make the false assumption that all magnitudes of the same kind are commensurable. But I count it one of the merits of the proof given above that it does not require us to deal with the question at all. Even if a "rigorous" argument is insisted on, it is easy to meet the requirement by prefacing the above proof with a few axioms embodying the properties of ratios without any reference to their measurement. Nevertheless something must be said somewhere about the measurement of ratios, and it may be conveniently said here and, perhaps, take the following form. If two quantities of the same kind, P and Q, contain respectively p and q units exactly, the ratio of their magnitudes is measured by the fraction p/q. But it is a rare thing to find a quantity which contains the unit an exact number of times, however small the unit may be. What we actually find is, as a rule, that P's magnitude is between p and $p + 1$ units, Q's between q and $q + 1$. In this case (which, I repeat, is the usual one) we cannot measure the ratio exactly; we can only say that it is between $(p + 1)/q$ and $p/(q + 1)$. If R and S are two other quantities, we can similarly determine that their ratio lies between $(r + 1)/s$ and $r/(s + 1)$. Now if there are any fractions which lie between the members both of the first pair and of the second, it is clearly possible that the two ratios *may* be the same;* and we shall conclude that they

*This would, for example, be the case if the first pair were 3·463 ... and 3·482 ... while the second pair were 3·471 ... and 3·502 ..., but not if the second pair were 3·483 ... and 3·496 ....

are the same if such fractions can always be found however small the unit is taken. This is, in fact, the only definition of equal ratios which can be practically applied. Having explained it, you may, if you please, go on to show that there are quantities whose magnitudes cannot, even conceivably, be both measured exactly in terms of the same unit; but the discussion would be a luxury which might well be postponed.

When the fundamental congruence theorems have been proved there is much to be said for going straight on to the corresponding similarity theorems, so that the whole substructure may be laid down before we proceed to build on it. On the other hand, it is important to separate clearly the properties which can be deduced by the congruence principle alone from those which are deducible only with the aid of the principle of similarity. Thus there is also something to be said for postponing the study of similarity for a while. But this is purely a question of expediency, which might affect the numbering of theorems (if it should be decided to give them official numbers), but could not affect their logical order.

The standard theorems deducible from the congruence theorems alone fall naturally into two groups. I should enunciate those of the first group in the following order: (i) If the rays AP and BQ, on the same side of AB, make the angles PAB and QBA together equal to two right angles, then the rays do not meet. (ii) The same may be proved if the angles are together greater than two right angles. (iii) If the lines do meet the angles are together less than two right angles. (*N.B.*—The converse cannot be proved.) (iv) The exterior angle of a triangle is greater than either of the interior opposite angles.

I choose this order because, as Hilbert suggests, (i) can be proved by direct application of the first congruence theorem and because the remaining proofs then become simpler than Euclid's. Moreover, the order seems "prettier" than his. It will, however, be shown below that the whole group can be deduced still more easily from the second similarity theorem, so that the only ground for taking it at this stage is to show how far the authority of the principle of congruence extends.

The second of the two groups consists of Euclid, 1. 18-20 and 24. These all depend on (iv) above and therefore could be postponed until siimlarity has been treated.

It is much more important for my purpose to formulate the standard theorems immediately derived from the fundamental similarity theorems; for, as I have already said, they are to include the doctrine of parallels. I suggest the following order. (i) If any transversal cuts two lines at the same angle those lines do not meet. (This is, of course, equivalent to (i) above.) (ii) On the same supposition, any transversal that intersects the former one also cuts the lines at the same angle. (iii) The angle-sum of a triangle is two right angles. (iv) On any line AB a triangle ABC can be drawn having the angles A, B and C respectively equal to any three whose sum is two right angles. (v) If two lines in a plane do not meet, any transversal must cut them both at the same angle. (vi) Through a given point there can be drawn only one line which will not meet a given line in the same plane with it. (Playfair's Axiom.)

I will give the proofs of (i) and (v) (Euclid, I. 27, 29). You may find it amusing to work out the others for yourselves.

(i) Let the transversal ABC cut the lines L and M in B and C, and suppose L and M to meet in N. Then the triangles ABN, ACN have two equal angles and are therefore similar. Hence $\angle ANB = \angle ANC$, which is impossible; so the lines cannot meet.

(v) Let $\angle ABL$ be greater than $\angle ACM$; then the interior angles CBL and BCM are together less than two right angles. Let the defect from two right angles be $\angle R$. Then by (iv) a triangle can be constructed on BC whose angles are respectively $\angle B$, $\angle C$ and $\angle R$. But this means that BL and CM must meet, which is contrary to the hypothesis. Hence $\angle ABL$ cannot be greater than $\angle ACM$, etc.

The substitution of a postulate of similarity for the parallel postulate is, as I have admitted, an audacious step. I am anxious, therefore, to plead for it the authority of great names. The first is John Wallis who, in seeking to "demonstrate the fifth postulate of Euclid," followed a method very different from mine but based upon the same assumption.* A much greater authority,

*It will be of interest to quote his actual words:: "Praesumo tandem . . . ut communem notionem.

Datae unicunque Figurae, Similem aliam cujuscunque magnitudinis possibilem esse.

Hoc enim (propter quantitates continuas in infinitum divisibles, pariter atque in infinitum augibles) videtur ipsa Quantitatis natura fluere; figuram scilicet quanlibet continue posse (retenta figurae specie) tam minui, tam augeri in infinitum." From *Opera* (1693), vol. ii., p. 674. I have followed up the references in Bonola's invaluable *Non-Euclidean Geometry* (Open Court Series).

Laplace, definitely held that a postulate of similarity is more natural than Euclid's parallel postulate, and considered this view to be confirmed by the remarkable fact that Nature appears to take no account of absolute size, but applies her mechanical laws indifferently to systems almost infinitely big and little. This argument has gained still more force since Laplace's day. Lastly, since this paper was read, Prof. Coulichere of Petrograd has kindly pointed out to me that W. K. Clifford, in his fragmentary but brilliant *Common Sense of the Exact Sciences,* has adumbrated a treatment of geometry similar in principle to the one here advocated. I find (I cannot candidly say with entire pleasure!) that he anticipated the proof I have just given of Euclid, I. 29.

SUGGESTIONS ON THE ARITHMETIC QUESTION

By DAVID EUGENE SMITH

The National Committe on Mathematical Requirements served, through its report, to stimulate inquiry on the part of those who know something about the problem of mathematics in the secondary school. The commission appointed by the College Entrance Examination Board, through its report, confirmed the important findings of the National Committee, and did much to eliminate the obsolete material in the high-school curriculum and to substitute therefor a more modern type of algebra, geometry, and trigonometry. It will take some time for schools and teachers to adjust the courses in mathematics to meet the recommendations of these bodies, to eliminate the over-drill, to cast out the useless part of the work in the elementary operations, and to realize that trigonometry is a part of algebra and that it can be made much simpler and more interesting than much of the drudgery (as the subject was commonly taught) that it replaces, but the leaven is working and the outcome will be on the right side. It takes time to develop the idea that we should seek quality instead of mere quantity, but our younger generation of teachers is coming rapidly to realize the significance of this idea in a subject, for example, like algebra. The reform would proceed more rapidly if it were not that nearly all of our current tests include a considerable amount of material that has been recommended for elimination by all who have given the subject serious thought. As Professor Upton has recently remarked in the MATHEMATICS TEACHER, schools often feel compelled to teach subjects that are obsolete, and possibly even to recognize forms that are incorrect, because of the carelessness shown in preparing many of these tests.

As to arithmetic, the question has not been discussed with anything like the care given to the mathematics of the secondary school. The psychologists have begun to give it some attention, but they approach the problem from only a single angle, and their conclusions as to what should be taught have generally been open to grave criticism. Even their conclusions as to how the special branches should be presented have, in general, shown an

unfortunate ignorance of the needs of practical life, these needs not infrequently tending to render invalid the reasoning which seems to characterize our educational laboratories.

It is proposed in this paper to consider two major topics in the hope that some representative committee may be prompted to investigate them in a broadminded and scholarly fashion.

First, as to certain phases of the operations taught to children; what ones are of such doubtful value to the great mass of our people as to make it probable that they should be eliminated or at least left for the technical training of the small number that may ever use them?

Second, what habits are we forcing upon children that will hinder them in the arithmetical operations which they must undertake in ordinary business life?

As to whole numbers, the four operations are probably taught to about the proper extent at present. Most textbooks published within the last few years have shown good judgment as to the extent of the work, whatever may be said as to the method of presenting it. As to the latter there is not space to speak in this article. The reader would find an interesting field, however, if he should examine all the suggestions made in the recent books upon the teaching of arithmetic. If he should take a certain writer seriously, he would feel that he had lost enormously in not practically memorizing considerably over a thousand number combinations,—a thing which not even an expert accountant would ever think of doing unless he were a freak.

Even with respect to simple addition we have of late been meeting with an interesting difficulty. Someone gets the idea that, because the result is written underneath, a child should be taught to add from the top down. This would, of course, be a legitimate conclusion if there were no other consideration. It must, however, be observed that the best psychological laboratory is the world itself, and before we attempt to change a universal custom of "adding up," we should consider the reason (we may quite properly speak of it as the psychological one) for its existence. Everyone who has done any computing knows the advantage of pointing to the figures when adding a column; it is almost a necessity,—certainly it is so if we wish to minimize the chance of error. He also knows the great gain in looking ahead

slightly so as to group his figures, a process that makes for both accuracy and speed. Now with these two facts in mind, let him try to add from the top down and he will observe that he cannot point with his pencil and at the same time look ahead so as to group his figures,—at least without an awkward position of the hand which the world will not accept. In other words, for a mere notion, not even based upon ordinary superficial observation to say nothing of business practice, and with not the slightest gain in accuracy, we find certain teachers advocating a plan which makes for inefficiency and which is not used and will not be used in the business world. It is allowable to add downwards as a check, and there is some psychological reason for so doing, although the process is a longer one and is probably no more conducive to accuracy; in fact, since it tends to lose the opportunity for grouping, it probably increases the chance of error.

If we should listen to certain other psychologists we might readily be led to feel that our brains have been unduly strained by placing the result in short division below the number divided, instead of above as in long division. We already see textbook writers forced to place it above because of three powerful influences,—(1) state and city requirements, (2) test papers in arithmetic, and (3) the authority of psychologists, who are no authority whatever on the general merits of the case. Anyone who has paid attention to the use that we are likely to have for short division knows that the placing of the result above the number divided is the most inconvenient way possible. It is so inconvenient in long division that its sole justification (and it is a perfectly good one) is that it serves to fix the position of the decimal point with less danger of error than arises from placing it, as formerly, to the right. Let anyone try finding $\frac{3}{4}$ of \$217.64; try finding the average of \$2.48, \$3.96, \$2.82, and \$3.24; try finding a discount of $\frac{1}{6}$ on a bill in which the items are \$3.27, \$9.03, and \$17.00,—these being the three leading types in which short division is used,—and let him see where he will naturally place the result. In any of these cases, if he places the result above, he must copy a number unnecessarily, and it is in copying that errors are as frequently found as in almost any other simple operation. The business world not only does not write the quotient above in short division, but it will not do so for the reason that it leads both to inconvenience and to inac-

curacy; and this in spite of the trivial argument that a pupil should not learn two forms, even though he will be compelled to learn another one if he is to use division in practical life.

There is another feature in division that demands attention, namely, the time that is devoted to the remainder. The use of this form came from the period before the decimal fraction was invented and when the common fraction was but little understood. It is somewhat of a necessary evil in teaching division, since this topic naturally comes before the decimal is introduced and before the child knows enough about common fractions to make much use of them. There is a type of problem in which it has a meaning by itself, as in knowing how many 3-cent stamps we can buy with 10c, and how much money is left over. But it is well known that, in any ordinary work of a non-technical character, the remainder plays no part. For this reason it is unfortunate that so many teachers are compelled to place so much stress upon cases involving remainders, and that they debate so seriously about the method of writing a number which the child never uses. It would be about as well if we agreed not to use examples in long division in which remainders occur, until we can teach the pupil how to divide dollars and cents, and then to tell him to neglect any remainder less than half the divisor. In short division we could teach a child to write a fraction in the result, this being more real to all of us than a remainder, and just as easily taught to children. The only reason why it is not generally given in textbooks is to be found in the dominance of outside influences.

As to common fractions, there has been a great improvement of late years, but much more remains to be done. This cannot be accomplished by textbook writers alone; they can go only a little in advance of the demands. It is the makers of state and city syllabi, those who frame examinations for state and city use, those who claim to speak for the psychologists, and those who publish arithmetic tests, who must help in the movement if it is to be a success, although it is just here, however, that the strongest reactionary movement is encountered. What has been accomplished by progressive teachers is to emphasize the use of halves, fourths, eighths, and sixteenths, and this has been a great gain. It is based upon the patent fact that, for the great mass of people, these are the fractions most commonly seen. The third

and the sixth have their uses, and even the seventh enters into the measure of the cylinder, but the emphasis has happily been shifted to the more common forms.

What has been lost sight of, however, is that although we need to use thirds, we may say, as a close approximation to the facts, that we never add halves and thirds, or eighths and sixths. Indeed, for the large majority of our people we could safely limit the addition and subtraction of fractions to halves, fourths, and eighths, leaving the rest to such special technical training as could be given easily when really needed in practical life, or else turning the work into a case of decimals, as would probably be done in practice. What is also lost sight of is that cases in which we need to add more than two fractions are so rare that we would be quite justified in limiting the work to three fractions at the most. If the argument is advanced that practical examples occasionally requiring the adding of four fractions, the answer might well be a counter question:—out of our 115,000,000 people, how many have probably had occasion to perform an operation like $\frac{2}{3} + \frac{3}{8} + \frac{7}{4} - \frac{3}{16}$ in all their lives? And yet, this is the very thing that many of our syllabi require, and that are found in certain examinations.

The case of multiplication is equally serious. As business customs exist today we have abundant use for a fractional multiplier. We need to find $\frac{2}{3}$ of \$1.50, let us say; to find the cost of $4\frac{3}{8}$ yards of silk at \$3.25 a yard; and even to find half of $7\frac{3}{8}$ yards. Such computations arise often enough to leave no doubt of their value to enough people to make the teaching worth while. But what about $\frac{2}{3}$ of $\frac{7}{8}$? $\frac{3}{4}$ of $2\frac{5}{8}$? and $2\frac{7}{8} \times 5\frac{2}{3}$? Where are they used and by how large a proportion of our people? If we can find real cases, as of course we can, shall we be as successful in finding a real need? That is, are the cases that we find numerous enough so that the "man in the street," the fairly well educated citizen, needs this kind of work? If he does, must he not relearn the process when the need arises, and is this not a simple matter?

Whatever may be the decision in this case, what can we say as to the division of fractions? We certainly need to be able to multiply \$2.50 by $\frac{3}{4}$, but do we need to divide it by $\frac{3}{4}$ or any other fraction? If we do, how about dividing $\frac{7}{8}$ by $\frac{2}{3}$? Do we

need it? What proportion of our people use it? If they need it, what hardship would it have been to learn it when the technical need arose, just as we learn how to drive an automobile? To carry the matter still further, how does the number of people of our acquaintance who ever had to divide $17\frac{3}{16}$ by $2\frac{4}{5}$ compare with the number who learned how to drive a car?

Now there is a considerable amount of important work with common fractions, and everyone should know how to perform it. What syllabus makers and psychologists and testers and arithmetic committees have not seemed to comprehend, however, is that the decimal fraction has displaced the harder parts of the operations with common fractions, and that much of the latter has, for most people, become obsolete. The textbook writers can do something to assist the movement, but no textbook could ever assist in the work if it eliminated all multiplication and division of mixed numbers like $3\frac{7}{8}$, because it would never sell enough copies to pay for the printing and never reach the schools that need the help. Such a book would be discarded for the reason that it advocated the very thing that most careful thinkers know is sure to come within the next generation, namely, a further simplification of the work to meet the real needs of our people, thus permitting of more drill upon the great essentials of computation.

The decimal fraction began to make a little headway nearly four centuries ago; it was fairly well known a century later; it began to have a firm footing when the new United States adopted the decimal system of money and France established the Metric System, and now the world at large uses it extensively. But we have not yet recognized that it has replaced the common fraction in a large field of work involving the four operations, and has rendered obsolete much that we still teach in our elementary schools; neither have we generally recognized the types of cases in which we actually use decimals, or the nature of the operations that are legitimate.

To take a simple illustration, the test papers of the present time generally give examples like the one here shown. Now it is evident that if this problem could arise at all it would arise in connection with measurements. Furthermore, the mere adding of five abstract numbers means nothing; it is useless except for drill, and

$$\begin{array}{r} 2.7 \\ 32.838 \\ 4.06 \\ 7. \\ .4 \\ \hline \end{array}$$

the drill should be upon forms that the pupil will use sometime and for some definite purpose. It is, moreover, a well-known principle of practical mensuration that the degree of accuracy in one measurement must be the same as that in every other one that is connected with the same pieces of work; that is, if we are measuring the length and width of a room, we do not measure the length to the nearest inch and the width to the nearest 0.001 of an inch; we measure both to the nearest inch, to the nearest half inch, or to any other unit agreed upon for both cases. Therefore, in the example above given, if the unit agreed upon is 0.001, the numbers should appear as here shown, the pupil should see them in no other form, and the principle above referred to should be stated to him, should be used in all his measurements of the given object, and should be recognized in all examples given him to solve. To give an unreal example and tell him to annex zeros so as to make it look real, is poor teaching and poor ethics. And yet it is precisely this which is forced upon textbook writers by teachers who must drill their pupils on examinations set by outside authorities, and in which this unreal type constantly appears. The examiners are either ignorant of the facts or else they say that the teachers wish this kind of work because it is in the books, and so we have a vicious circle. The only hope would seem to lie in the recommendations of some representative body with vision enough to see the difficulty, with courage enough to attack it, with judicial power enough to decide correctly upon a set of recommendations that will bear scrutiny, and with scholarly standing enough to have its conclusions accepted.

2.700
32.838
4.060
7.000
.400

The same difficulty arises in many other cases at the present time. Children are asked to find the area of a rectangle 2.7 in. by 3.42 in., a case that is unreal for the reasons specified. They are required to find to four decimal places the circumference of a circle whose diameter has been found by measurement to be 4.3 in., a case that is unreal because the result cannot have a higher degree of accuracy than the data, and hence should not be carried beyond the nearest 0.1 in. They are told to find the average length of the sides of a triangle which, by measurement, are found to be 2.4 in., 2.7 in., and 2.2 in. and they give it as 2.4333 +, which is unreal for reasons stated above. The general question of degree of accuracy is, of course, beyond the under-

standing of children, but the simple principles laid down above should govern at least the making of proper syllabi in a state or a city.

Such are a few of the problems that relate to the arithmetic situation. They are only types, and they touch merely on a few details of the work with integers and common fractions, and on a single one connected with decimals. There are many others, some of which are even more serious, but they serve to give some idea of the nature of the questions which should be more carefully considered by scholars in these particular fields than has yet been the case.

HIGH SCHOOL MATHEMATICS CLUBS

ZULU REED
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In most of our high schools our courses of study are too crowded with formal work to permit our following, in the class room, interesting by-ways which touch our paths. Such is the case in mathematics. It is largely because of the failure of teachers to provide opportunity for exploring these mysterious by-ways that many high school pupils regard mathematics as a necessary evil, as something almost entirely apart from every day life, as an affliction to be escaped as soon as the minimum requirement has been met. But work that has been considered drudgery may become a pleasurable pastime with the proper incentive. Some one has fittingly said that the greatest problem to be solved by the mathematics teacher today is the problem of making mathematics interesting to the pupil. The mathematics club offers a means of helping to solve this problem.

The purpose of the mathematics club, therefore, is not merely to teach more formulas and equations and theorems than can be crowded into the class periods. It is to allow the pupil to explore and to discover for himself the many interesting and worthwhile pleasures that are to be found there. Through the club the pupil comes to know that the queer-looking portraits, which he may have seen in his algebra text, represent human beings not greatly unlike himself. His admiration for these ancient mathematicians grows as he realizes how much we owe to them for their valuable discoveries. He learns of the struggles the science of mathematics has passed through as it has come down through the ages. His imagination is stimulated as he lives, in mind, with the inhabitants of "Flatland." His enthusiasm is perhaps at its highest when he is skillfully playing a mathematical trick to the amazement of his club friends, or when he suddenly sees the fallacy in such a proof (?) as 2 equals 1. That the mathematics club has a real function in the high school, there is no doubt.

In writing concerning a mathematics club in a girls' school in England, Miss Irene M. Brown says that the reason the old,

yet ever new, question, "What good do I get out of all of this?" is so frequently asked of mathematics teachers by their pupils, is because the lighter side of mathematics is so seldom put before them. Also, it is easy to show that other school subjects, such as history, modern language, botany, English, make substantial addition to our pleasure in life. But when mathematics is brought into touch with every day affairs, its connection is usually financial, or at least strictly useful. Mr. C. W. Newhall of Shattuck School, Fairbault, Minn., in discussing the work of his club and some of the difficult subjects that its members chose to study says, "Even if the students do not acquire perfectly clear notions about these rather vague subjects, it does no harm at least to set them thinking. This is perhaps the most valuable result of the work of our club. It sets the boys to thinking and stimulates their imagination. They find that there is more of interest in the dry subject of mathematics than they had dreamed. Our investigations are not very profound to be sure, and will result in no contributions to the sum of mathematical knowledge, but the boys enjoy the meetings, and I think profit by them, and as these are two distinct objects of this mathematics club, I feel that it has proved itself worthwhile." Mr. F. C. Gegenheimer of the Marion, Ohio, High School writes that in his school the mathematics club is answering the question, "What shall we do for the bright students?" Miss Beulah I. Shoesmith, of the Hyde Park High School, Chicago, Illinois, in an article on mathematics clubs reports that the study of algebraic fallacies in her club created such an interest that pupils outside the club asked to be allowed to visit the club meetings; and that so much enthusiasm was aroused over the working out of original proofs for the Pythagorean Theorem by the advanced pupils, that the plane geometry students began to inquire when they would be allowed to study this entertaining theorem. Such lively interest as is reported from these clubs is the need of every high school mathematics department.

Why are not high school mathematics clubs more commonly found today? There seem to be three chief reasons. First, teachers lack a vision either of the need or of the possibilities of such an organization. It is much easier to do things year after year in the same old way than to be progressive. And so we, human beings that we are, choose the path of least resistance and

close our eyes to the fact that our pupils are suffering as a result. As we face the question squarely, we must admit that it is surprising that as many pupils as do elect advanced mathematics. Certainly it is often done in spite of our poor efforts rather than because of any stimulation or inspiration they have received from us. We fail to see the possibilities of a mathematics club because we know very little more about our subject than we find recorded in our text books. It is only personal study that can solve this problem—study that we owe to ourselves and especially to the young minds entrusted to us. Second, some teachers are unwilling to assume the responsibility that comes with sponsoring such an undertaking. Surely though, this group is in the minority. The general tendency is for conscientious teachers to allow themselves to be overburdened with responsibilities. True, all teachers are not suited to positions as club sponsors. But though each of us can't be a leader, there is just as much virtue in being a good follower. So, where there is more than one mathematics teacher in the school, the teacher who is not cut out for a club adviser, may be just as useful as a willing assistant to the adviser. Third, is the handicap, in many schools, of having no library equipment for the mathematics department. Perhaps at times we mathematics teachers are prone to complain that the high school libraries are filled principally with English and history references, with no provision made for our work. But, after all, is it not largely our own fault? We have made very little or no demand for library equipment. The teacher who makes her plea sufficiently strong and specific generally gets her wants fulfilled. However, a large library is not necessary for a beginning. The following books, together with the Mathematics Teacher and School Science and Mathematics, would make an excellent beginning library:

Smith, D. E.: *History of Mathematics, Volume* 2. Ginn & Co., 1925; price $4.40.

Jones: *Mathematical Wrinkles.* (Consisting of knotty problems, mathematical recreations, answers and solutions; rules of mensuration, short methods; helps, tables, etc. S. I. Jones, Gunther, Texas, 1912; price $2.10.

Ball, W. W. R.: *Mathematical Recreations.* Tenth Edition. Macmillan, $3.50.

Abbott: *Flatland.* (A. Square, an inhabitant of Flatland,

tells of the life in his world of two dimensions. Little Brown & Co., Boston, 1915; price $1.25.

Weeks, R. M.: *Boy's Own Arithmetic.* E. P. Dutton; $2.00. An interesting satire on traditional methods of teaching arithmetic.

Below are some suggestions for club programs:

I. THE SLIDE RULE

The following story makes an interesting introduction to the advantages of the slide rule. It appeared in the Mathematics Teacher, Oct., 1923; Vol. 16; pp. 332-334.)

TIME IS MONEY

Bill and Jack are two high school boys and both are members of the baseball team. Jack is the star pitcher for the school and Bill plays third base. A few days ago after school these boys met and the following conversation took place:

"I hear you have another raise in salary, Jack; you certainly are in luck."

"No luck about it, Bill. I'll tell you how it happened. You know I have been working after school three days a week in the office of an electrical manufacturing firm. Yesterday while I was in the office, our firm had a call on the 'phone from a contractor asking for an estimate on a big order of goods running into thousands of dollars. The contractor wanted an answer while he held the wire or he would give the order to some one else. My boss didn't know what to do about it. He couldn't figure out a reasonable estimate in so short a time and was going to tell the contractor so. But I thought of my slide rule and I said, 'We can do it. Just give me the price of each article and the number to be ordered. My slide rule will do the rest.'

"Well in less time than it takes to tell the story, we had the estimate ready and had closed the contract.

"You see the slide rule did all the multiplication instantly. If it hadn't been for that, we would have lost the order. Of course my boss was pleased and that's how I happened to get a raise in salary."

"Good work, Jack, but tell me, what is the slide rule like? I have never seen one."

"The slide rule is just a little piece of wood about a foot long that can easily be carried in your pocket. It looks something like a twelve-inch ruler.

But come out on the field, Bill; I want to show you a new out-drop I've been practicing and am going to use in our next game against Eastern High School. It certainly is a winner."

"Nothing doing. I have 20 examples in physics to do for home work tomorrow and, believe me, they are some work."

"I have the same lesson, Bill, but it is all done. I just ran those problems off on my slide rule. You know almost every problem in physics comes out in proportion where you have to multiply two numbers together and divide by a third number. The slide rule does all this for you and gives the answer to any proportion in a few seconds.

"And speaking of baseball, our scorer fixes up the batting average of every man on the team by using this same little instrument. You see all you need to do is to divide the number of hits by the number of times at bat and get the result correct to three figures. This is done instantly on the slide rule. It certainly is the quickest calculating machine I have ever seen.

"I'll tell you another thing, Bill; I use this rule in my trigonometry home work. It helps a fellow to understand the subject. I never understood what it meant to add logarithms till I saw it done on my slide rule. You see after solving a triangle by the use of tables, I can find out right away by the rule whether the answers are correct. If not correct, the slide rule tells me where the error is. This is why I always get my work right and have it done in half the time it takes the rest of you fellows."

"Say, Jack, that instrument must be what my sister was talking about last night. You know she is private secretary to the president of the Northern Electric Co. She said she checked the reports that came to her office yesterday in one-third of the time that it usually took and she did it with a slide rule.

"But how about using this rule in the New York State Regents' examinations, can we do that, Jack?"

"We certainly can. At the top of an examination paper that was given last year it says, 'The use of the slide rule is allowed for checking.'

"Here comes my big brother, Henry. Say Henry, what do you know about the slide rule? Did you ever use it while you were in war?"

"The slide rule. I should say so. Do you know that if it hadn't been for the war, you fellows would probably never have heard about this instrument. I learned to use this slide rule while recovering from injuries in a government hospital at Camp Grant. While lying in bed, it was fun to fool around with the rule and see what a lot of calculations I could do in quick time. When the war closed some people thought that an instrument that could be taught to soldiers in the hospital and that would save so much time ought to be used in the schools. So it was recommended for high schools by a national committee and that is how you came to know about it. I hear that it is coming into use in the regular mathematics classes in the high schools and colleges as fast as teachers and students learn how interesting it is and how much time it saves. Why, even your kid brother, Tom, came home from his Junior High School today and said his class had been using the slide rule. What do you think of that, Bill?"

"I'll say it is mighty interesting, Henry. Wouldn't it be great if our school should teach us more about the slide rule? Look at the home work we have and think of the time we would save in our mathematics and physics. Don't you say so, Jack?"

"That's all right, Bill, but haven't you heard the news? The principal has ordered some slide rules for the school. Everyone in the commercial department, in trigometry, and in intermediate algebra is going to be taught how to use the rule. As for me, my dad gave me a slide rule for my birthday. A self-teaching manual that anyone can understand came with it. I worked up the use of the slide rule by myself."

"That sounds good to me, Jack, and I'll tell you what I'm going to do. Tomorrow morning, bright and early, you will find me the owner of a slide rule like yours."

If desired a slide rule, accompanied by a self-teaching manual, may be bought from the Keuffel and Esser Co., New York, Chicago, St. Louis, San Francisco, and Montreal. A very satisfactory one is the Mannheim Slide Rule, price $4.00, which is accompanied by a Self-Teaching Manual with tables of settings, equivalents and gauge points, by Wm. E. Breckenridge of Teachers College, Columbia University. However, from the same company, student slide rules may be bought for $1.00, and up.

It will perhaps be financially impossible for each pupil to buy a slide rule. So here are instructions whereby each pupil may make his own, taken from "A Slide Rule Constructed Without the Use of Logarithms," by A. H. Fensholt, as referred to in the bibliography.

For Constructing a Slide Rule: The construction requires a strip of wood or paper, which will be known as the "main scale," and a smaller piece of wood or paper known as the "index piece." Upon the index piece is marked a vertical line marked 1, and at any convenient distance to the right of this line is placed another vertical line, marked 2. (Fig. 1.) The distance between 1 and 2 is immaterial.

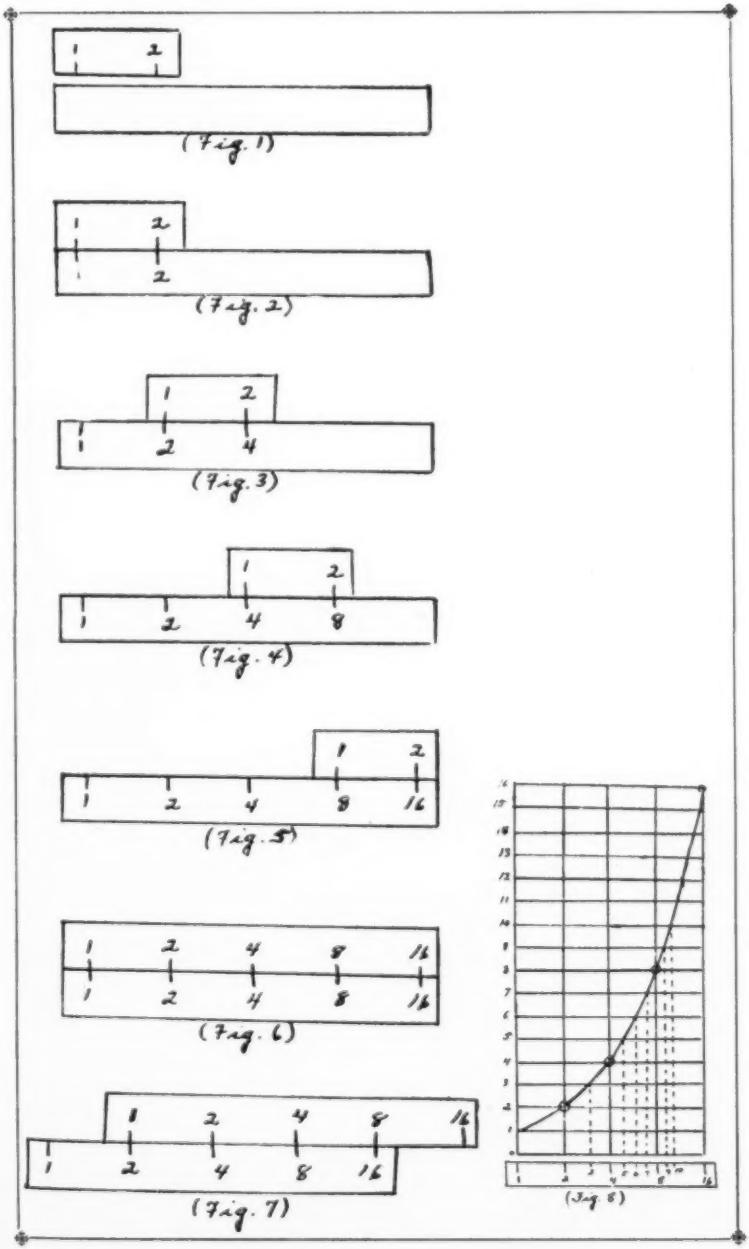
The index piece is then placed parallel with the main scale at one end (Fig. 2) and marks on the index piece are referred to the main scale.

The index piece is then moved to the right until number 1 of the index coincides with number 2 of the main scale. (Fig 3.) It is then possible to multiply 2 on the main scale by 2 on the index. Therefore under number 2 of the index should be placed the product, or 4.

By moving the index piece again until number 1 of the index now coincides with number 4 of the main scale, it is possible to multiply 4 on the main scale by 2 on the index. (Fig. 4.) Therefore under number 2 of the index is placed the product or 8. Similarly, for figure 5.

It is evident that so far the instrument can be used only for multiplying the numbers 1, 2, 4, and 8 by 2. If the index piece is discarded and a new piece of wood substituted as long as the main scale, all the numbers may be transferred to the new strip. (Fig. 6.) The two scales upper and lower are now identical.

By moving the upper scale (Fig. 7) until number 1 of the upper scale coincides with number 2 of the lower, it is possible to multiply with this setting of the rule, 2 by 2, 4, or 8. By moving the upper scale until number 1 of the upper scale coincides with number 4 of the lower, it is possible to multiply 4 by 2 or 4. The rule is still practically useless however because numbers 3, 5, 6, 7, 9, and 10 are missing. The question is as to the exact position of these numbers. It is unnecessary to go higher


1 2
(Fig. 1)
1 2
1 2
(Fig. 2)
1 2
1 2 4
(Fig. 3)
1 2
1 2 4 8
(Fig. 4)
1 2
1 2 4 8 16
(Fig. 5)
1 2 4 8 16
1 2 4 8 16
(Fig. 6)
1 2 4 8 16
1 2 4 8 16
(Fig. 7)
(Fig. 8)

than 10, and 16 will be dropped as soon as the position of 10 has been definitely fixed.

In figure 8 the distances between 1, 2, 4, 8, 16 are all equal. The position of numbers 3, 5, 6, 7, 9, 10 are still to be found.

Above the lower scale and parallel with it are drawn horizontal lines, spaced equal distances apart and marked 0, 1, 2, to 16 inclusive. Vertical lines are drawn from the lower scale at numbers 1, 2, 4, 8, and 16. The points at which the vertical lines of a certain number cross the horizontal lines of the same number, as for instance, where vertical line number 4 crosses the horizontal line number 4, are marked by circles. Through the five circles thus located is drawn a smooth continuous curve. It is evident that the curve thus drawn crosses all horizontal lines from 1 to 16 inclusive. It is very easy, therefore, to find the point at which the curve crosses the horizontal line number 3. Exactly below this point is the position for the figure 3 on the main scale. The positions of 5, 6, 7, 9, 10 are found in a similar way.

In order to construct the complete slide rule scale, it is also necessary to have the fractional divisions between the numbers, such as 1.1, 1.2, 1.3, etc. All these numbers can be located in the same manner as the whole numbers 3, 5, 6, etc. The scale may be subdivided as finely as desired by continuing the process of finding the points on the curve, and dropping these points to the main scale below.

II. TRICKS AND PUZZLE PROBLEMS

Tricks and puzzles have always been of interest to the human race. It is never difficult to interest students in making a collection of them. Below are a few that may be used.

A TRICK WITH CARDS

By R. M. MATHEWS
Riverside, California
School Sci. and Math., Dec. 1913; Vol. 13; p. 820.

The performer presents the company with a pack of playing cards and instructs them to make, in his absence, as many piles as they please as follows: The spots on the bottom card plus the number of other cards shall make 12. Thus, if the bottom card is 8 then four other arbitrary cards are to be laid on it to

make up the pile. Every face card is to count as 11. The number of piles and the bottom card of each pile are arbitrary. On returning the performer receives the remaining cards and at once tells the sum of the spots on the bottom cards of the piles. What is the formula for performing this trick?

Let s denote the required sum of the spots, n the number of piles, and r the number of cards left over. In each pile the number of spots on the bottom card plus the number of cards (bottom card included) is 13. So therefore, $13n$ plus r equals s plus 52.

$$s \text{ equals } 13\ (n-4) + r \qquad [\text{or, } 13n + r = s + 52 \quad s = 13\ (n-4) + r]$$

To conceal the fact that he must count the remainder, the performer resorts to various devices, such as requesting one of the company to surround the piles with a "magic circle" of the remaining cards, or to deal out the remainder to the piles.

A TRICK WITH NUMBERS

(From *Number Games and Number Rhymes,* by D. E. Smith and Students; Teachers College Record, Nov., 1912; Vol. 13; pp. 385-494.)

Have some one think of a number. Then tell him to add 5, multiply the sum obtained by 3, and subtract 7 from the product.

The original number thought of can be found by subtracting 8 from the result and then dividing by 3.

Representing the number thought of by a and the result of the above operations by p, we have,

$$\begin{aligned} (a+5)\,3 - 7 &= p \\ 3a + 15 - 7 &= p \\ 3a + 8 &= p \\ a &= \frac{p-8}{3} \end{aligned}$$

For example, suppose the number thought of is 6. Then we obtain the result 26, whereupon the original number thought of is obtained by

$$26 - 8 = 18; \quad 18 \div 3 = 6.$$

Students may be allowed to work out the formula for themselves, and also to originate similar tricks.

SOME INTERESTING TABLES

(Also found in, *Number Games and Number Rhymes*, by D. E. Smith and Students.)

$$
\begin{aligned}
9 \times 9 + 7 &= 88 \\
98 \times 9 + 6 &= 888 \\
987 \times 9 + 5 &= 8888 \\
9876 \times 9 + 4 &= 88888 \\
98765 \times 9 + 3 &= 888888 \\
987654 \times 9 + 2 &= 8888888 \\
9876543 \times 9 + 1 &= 88888888 \\
98765432 \times 9 + 0 &= 888888888
\end{aligned}
$$

$$
\begin{aligned}
1 \times 1 &= 1 \\
11 \times 11 &= 121 \\
111 \times 111 &= 12321 \\
1111 \times 1111 &= 1234321 \\
11111 \times 11111 &= 123454321 \\
111111 \times 111111 &= 12345654321 \\
1111111 \times 1111111 &= 1234567654321 \\
11111111 \times 11111111 &= 123456787654321 \\
111111111 \times 111111111 &= 12345678987654321
\end{aligned}
$$

$$
\begin{aligned}
1 \times 8 + 1 &= 9 \\
11 \times 8 + 11 &= 99 \\
111 \times 8 + 111 &= 999 \\
1111 \times 8 + 1111 &= 9999 \\
11111 \times 8 + 11111 &= 99999 \\
111111 \times 8 + 111111 &= 999999 \\
1111111 \times 8 + 1111111 &= 9999999 \\
11111111 \times 8 + 11111111 &= 99999999 \\
111111111 \times 8 + 111111111 &= 999999999
\end{aligned}
$$

There was a time when problems appearing in rhyme were very popular. These are some typical examples, most of which appear in the Epigrams of Metrodorus about 500 A. D. They are found in *Number Games and Number Rhymes*, by D. E. Smith and Students; Teachers College Record, Nov., 1912; Vol. 13; pp. 385-494.

THE STUDENTS OF PYTHAGORAS

"Pythagora, on you I call,
How many study in your hall?"

"Polyc'rates, oh listen well!
One-half of those who with me dwell
Learn science; and a quarter part
Of my disciples learn the art
The Sophists teach; while listening mute,
One-seventh sit through all dispute.
Three studious girls have I beside,
How many, then, with me abide?"

THE ALLOYS IN THE CROWN

"A crown of copper and of gold
With tin and iron firmly rolled
The goldsmiths shall prepare for me
Of sixty minae's weight, to be
Three-fourths of tin and gold. They must,
Moreover, have a care that just
Two-thirds shall be of gold and copper
And then shall mix the metals proper
To have of iron and of gold
Just three-fifths." Now the story's told,
And you, in turn, shall write it down
How much of each was in the crown.

THE PIPES AND THE CISTERN

It takes one day to fill the vat
With this large pipe; two days with that;
The third pipe needs but one day more;
The fourth pipe fills the vat in four.
If all four pipes together run,
How long before the task is done?

A PROBLEM ATTRIBUTED TO EUCLID

"If of your coins you give me two,
Then I'll have twice as much as you."
"O Friend, if two thou give'st to me
I'll have four times what's left to thee."
Read me this riddle now my lad,
And tell me how much each one had.

A PROBLEM OF THREE NUMBERS

"I wish three numbers now to find;
And these must be of such a kind,
The first shall be the second plus
One-third the last. The second thus
Is found: One-third the first you take,
And to this add the third. You make
The third by adding on to ten
One-third the first. This given, when
You've worked it out, come tell to me
That which I ask,—the numbers three."

THE PROBLEM OF THE GOLDEN VASE

Ingenious Craftsman, show your art!
If you should take a quarter part
Of this rare vase of pure gold,
And melt and pour it in a mold,
Together with a third and twelfth,
The sum of all this molten wealth
Would weigh one mina. What, I pray,
Does the golden vase at present weigh?

A PROBLEM OF TIME

"What is the hour, comrade, pray?"
"At a dozen hours to the day
The time that's past since the day begun
Is three-fourths that to the set of sun."

Now quickly scan this simple rhyme
And tell me what's the proper time.

A PROBLEM OF AGES

When first the marriage-knot was tied
Between my wife and me,
My age did hers as far exceed
As three times three does three;
But when ten years, and half ten years,
We man and wife had been,
Her age came up as near to mine
As eight is to sixteen.
Now, tell me, I pray,
What were our ages on the wedding day?

"Perhaps the culmination of this rythm method may have been the efforts of Master Capen, in the old Mayhew School of Boston, early in the nineteenth century, who had the boys sing to the tune of Yankee Doodle:

Five times one are five,
Five times two are ten,
Five times three are fifteen,
And five times four are twenty." D. E. Smith.

3. SONGS, CONTESTS, GAMES

These two songs are from an article on mathematics clubs by Louisa M. Webster, in the Mathematics Teacher, June, 1917; Vol. 9; pp. 203-208.

O DEAR, WHAT CAN THE MATTER BE?

(Tune, old English folk song by the same name)

O dear! What can the matter be?
Dear, dear! what can the matter be?
O Dear! what can the matter be?
This problem won't work out right!

We've struggled, we've juggled, equations we've buggled;
We've added, divided, subtracted,—we tried it
Most sweetly, discreetly, and then we decided
To let it severely alone.

O dear! what can the matter be?
Dear, dear! what can the matter be?
O dear! what can the matter be?
The ghost of the problem remains.
It haunts us, it taunts us, with errors it flaunts us;
We try to efface it, but mem'ry won't chase it—
When, lo, inspiration!
With courage we face it, and quickly we make it our own.

NUMBER SONG

(Tune—"Lula is Our Darling Pride.")

Numbers are our pride and joy,
Numbers great, numbers small;
Juggle with them like a toy
At our beck and call;
Nor does vast infinity
Puzzle or distress us;
Nor can zero (hard to see)
In the least oppress us!
Numbers are our pride and joy,
Numbers great, numbers small;
Juggle with them like a toy
At our beck and call.

The following games are from Number Games and Number Rhymes, by D. E. Smith and Students; Teachers College Record, Nov., 1912, Vol. 13; pp. 385-494.

LAWYER

The players are seated in two rows facing each other, those opposite being partners. The leader of the game is called the Lawyer. The Lawyer asks a question of a player and the player's partner answers immediately in the same form as if the question were addressed to him. For example, the Lawyer asks "Can you tell me how much five times eight are?" and the partner of the one questioned must answer without hesitation, "Yes, I can tell you that five times eight are forty." If the one addressed answers, he becomes the Lawyer. If the partner makes a mistake in his answer, he becomes the Lawyer. In case both make mistakes, the Lawyer decides who is to take his place. (This is adapted from the parlor game of the same name.)

TWENTY-ONE

This game requires four sets of cards numbered from 1 to 8 for from four to six players. The cards are shuffled thoroughly and are then spread face downward on the table.

The object of the game is to hold cards that add up to twenty-one. One point is scored for the player who holds cards whose sum is nearest to twenty-one, and two points are scored for the one whose cards add exactly to twenty-one. No score is allowed for a number in excess of twenty-one. The

game may be decided by the highest score made, or by 5, 10, or any number previously decided upon.

Each player draws in turn. When all have drawn as many cards as they wish, the hands are laid face upward on the table and are counted. Anyone announcing a wrong sum forfeits a point. When the cards are exhausted, they are re-dealt in case the game is being played for a definite number of points.

NUMBER BUILDING

For four, five, or six players, prepare four sets of small cardboard squares on which are written the numbers used in the multiplication tables.

The numbered squares are placed in the center of the table, face downward. Each player draws a square and the one with the lowest number leads. The leader draws a number and places it face upward in front of him. The next player does likewise, and this is continued until someone draws a number which is the product of two other numbers in sight. The player who has drawn this number, takes all three numbers and places them in a pile with the product number on top. The play continues until a player has ten different products. (This is adapted from the word game of "Logomachy.")

RING TOSS

Rings of different colors and sizes are thrown at a standard. The rings have different values and those that encircle the standard are added for the score. The group of players may be divided into teams, each team throwing a certain number of rings; or each team may be allowed to play for a certain number of minutes.

This little poem may be used as a contest, individuals or groups competing in discovering and underlining the greatest number of mathematical terms:

LOVE MATHEMATICAL

(By Captain Robert C. Gillies, Mathematics Teacher, May, 1920; Vol. 12; pp. 124-125.)

A was a *constant* of the sterner sex
Though somewhat ar*bit*rary:
He burned with a passion for the sweet Miss X
Who often proved contrary.
She was a vari*ab*le, need we say?
And very independent;
Though wierd was her lo*cus*, always A
Was her faithful, sure attendant.

Now, A had a rival by the name of Y,
Whose actions were distressing;
Miss X he seemed to *be governed by,*
Respect to her confessing.
Often a radical but mostly square
He wooed her with great unction:
So that love between A and his lady fair
Was a dis*contin*uous func*tion*.

A had two friends, young B and C,
To whom for consolation
He went in his great anxiety
To solve this hard equation.
"I don't like the sound of $XY + A$,
$AX - Y$ would be better;
To eliminate Y I have tried all day,
Confound the pesky letter."

No way could B see, but C happened to be
Of guile a great exponent;
"It is not complex to annex Miss X
And to vanish your opponent.
Get her to marry you, if not today,
Next time you have a chance, sir;
Y vanishes, you know, when $X = A$
So there you have your answer."

B added, "Yes, that is the rational way,
If we must be explicit;
Just do as we say and implicit obey
And success? You cannot miss it.
To have a sweet mate is a happy state,
Celibacy is private;
When wed you integrate and co-ordinate
And joy's the first derivative."

A granted that C's was a clever ruse;
"You're an acute observer;
I may be obtuse, but there is no use,
I love her with true fervor;
She's of normal form and great symmetry,
But this is my position:
The witch is irrational, Q. E. D.
I've proved my proposition."

The object in the following contest is to write after each of the statements a geometrical term suggested.

PUNS FROM GEOMETRY

1—A flattering remark. (complement)
2—That which Noah built. (arc)
3—A harmless kind of worm. (angle)
4—A special kind of meter. (diameter)
5—What the bloodhounds do in chasing down a woman criminal. (center)
6—A part of John the Baptist's diet. (locus)
7—A gram but not a measure of weight. (parallelogram)
8—A bus, but not a taxi. (rhombus)
9—A measure of temperature. (degree)
10—A process of making a rough board smooth. (plane)
11—An expression to represent the absence or loss of a parrot. (polygon)
12—An angle that will not give up. (triangle)
13—An appropriate title for a knight by the name of Kell. (circle)
14—An article for serving picnic ice cream. (cone)
15—The state of blindness. (secant)
16—A blending of musical tones. (chord)

17—That which may affect one's heart upon climbing Pike's Peak. (altitude)
18—An appropriate name for a show with Mr. Ray as manager. (ratio)
19—A tall coffee pot in use. (hypotenuse)
20—If to "are" you add a letter. (area)

4. A pageant suitable for May Day; appearing in the School Science and Mathematics, May, 1920, Vol. 20; pp. 457-459.

THE MATH QUEST

(By Helen Whitaker, Topeka, Kansas)

Cast of Characters: Algebra; The Alphabet, twenty-six girls. Ten X's, ten girls. Three —X's, three boys. Numbers 0 to 9, ten boys wearing crowns with numbers in front.

Scene—An opening in the woods.

(Algebra enters.)

Algebra—In sooth, I know not what the number is,
That subtle number, it eludes me still,
And how to catch it, find it, or come by it,
I know not. With this care I am distraught;
I've searched the woods to find the answer there.
I've peered in books for method, hint or guide,
And every disappointment makes me fear
Fortune doth mock my venture and withhold
The mystic number that the Queen demands.
"Bring me the number, ere again you come,
Whose square, when lessened by its seventh sum,
Is then worth naught." And I have failed:
But no, there's someone else I'll ask.
I'll call my letters forth to aid the task.

(The Alphabet enters at her call)

O, Alphabet, when last from all our realm
We met to crown our Queen, the Queen of May,
She gave to all her subjects some great task
To do for her. That was a year ago.
And now today I fain would greet her court,
But am forbid—my task's not finished yet.
For this I seek your aid. Thus spake the Queen,
"Find me the number, ere again you come,
Whose square when lessened by its seventh sum
Is then worth naught." Now hasten to the task,
And make all speed to find me what I ask.

(Letters march and form a figure 8, kneeling on the grass.)

'Tis wrong, what you suggest.
"8" will not end our quest.

(They try again, this time forming a four.)

That's not the one I need,
But one more trial—proceed.

(Letters form a 10)

Nay, nay, that will not do. It is not right.
'Tis not the number that the Queen desires.
But thou, fair X, art wiser than the rest—
I beg you stay.
The rest away.
I have no further need of you today.

(Alphabet exeunt, all but X.)

Now X, bring aid.
On you is laid
The task the Queen did ask of me.
"Find me the number—seek it with great care—
Whose seventh sum, while it makes less the square,
Doth equal naught." This, the Decree.

(While she speaks, the ten other X's enter. They come quickly up to X, and she joins herself to them.)

These ten strange X's joined to mine I see
Have made the sum of X's eleven, for me.

(Enter three —X's. They are joined by three of the girls, and wander off together.)

These minus X's through an adverse fate
Have changed the total number back to eight.
Alas, 'tis ever thus with girls in Math.
My problems and my theory they forsake—
When men are near. 'Tis many an hour they waste.
In talking, roaming care free through the wood.
As to the rest, my faithful eight, to you
I give the task: to find the answer true.

(Enter the Number System.)

Welcome, O Number System, to our land.
Whence come ye, and what seek ye at our hand?

Zero—We come from far, in search of lady fair,
One X, whose name is honored in the realm
Of Math, wherever Algebra is known.
Under her rule, her subjects would we be.

Algebra—Yonder she stands, my friend and helper, she.

(Number System leads X to one side, 2 puts his crown on her, they help her onto a log, and all hail her as queen.)

Algebra—Look now, what do I see? X raised to a power.
The numbers crown her, seek her for their Queen
And place upon her head their crown of two,
The index of her power. By a log
They've raised her thus above the other X's.
She's now the square of what before she was.
The square—the SQUARE—why now I see—
She is the first part of the Queen's decree.
And for the rest, the sum of X's left
Is seven. If this be naught, my X is the one
Whose square when lessened by its seventh sum
Is then worth naught.

Zero—I am but naught; if any help I'd be
In your great search, I'm ready for the task
And I'll be equal to what e'er you ask.

Algebra—Come all, according to the Queen's decree
Let me find out what number X may be.

(They form the equation: $X^2 - 7X = 0$, with X first, then the seven X's in a row and last Zero, holding the minus and equality signs in their proper places.)

Algebra—X square minus seven X is naught. Yes, true.
When this is solved, my task indeed is through.

(She waves her wand. In the confusion following, X is uncrowned. She takes Zero and Seven by the hand and brings them to Algebra.)

Algebra—My problem is solved. Two answers do you bring—
Seven and Naught. Yes, I receive them both:
For both fulfill the Queen's desire of one
Whose square, when lessened by its seventh sum
Is then worth naught.
Rejoice with me, my task is now complete.
In joy and honor to the court we'll go,
And there these wise solutions proudly will we show.
Come one, come all,
Away, away,
To the court of our Queen,
Our fair Queen of May.

5. A hike into the woods to study mathematics in nature.

The club may be divided into two or more teams for a contest as to which can discover the greatest variety of geometric forms.

6. A Study of the Abacus.

D. E. Smith's *History of Mathematics*, Vol 2.

(a) The dust table.
(b) The wax tablet.
(c) The line abacus.
(d) The slate.
(e) The abacus of Rome, China, Japan, Russia.

Models of these may be constructed by the pupils. A very interesting contest may be made of addition and subtraction on the line abacus.

7. Ancient Numerals of Various Countries.

D. E. Smith's *History of Mathematics*, Vol 2.

(a) Babylonian.
(b) Chinese.
(c) The Maya.
(d) Egyptian.
(e) Greek.
(f) Hebrew.
(g) Roman.

8. The Story of Dr. Abbott's "Flatland" (this has been very successfully dramatized by some clubs) followed by

9. A Discussion of the Fourth Dimension.

10. Mathematics and Religion, as inspired by Dr. D. E. Smith's "Religio Mathematici" referred to in the bibliography.

11. Debate: Resolved that the Metric System should be adopted as the single standard of measurement in the United States. (Some references are given in the bibliography. Additional ones may be found in the article referred to in the bibliography by Mr. A. H. Wheeler on Mathematics Clubs.)

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REAL IMPROVEMENT IN ALGEBRA TEACHING

By HARRY C. BARBER
English High School, Boston

I.

You are experts in education. You are experts in that part of education which is concerned with the teaching of mathematics. As such I have a question to ask you.

If I send to you my fourteen-year-old daughter for forty minutes a day throughout the school year, just what do you expect to be able to accomplish for her? What is to be the result of these daily visits in your class. Both she and I are very much interested in your answer.

Perhaps you want to know if I expect her to be prepared for college. I do not know whether she will go to college, neither can I assist you by telling for what particular vocation I want her to be prepared. We have not yet been able to look so far ahead. But there she is at the door of your class room. She is to be in your presence forty minutes a day for perhaps 180 days. What is to be the outcome?

I very well remember when that question first presented itself to my mind as a child. I was in the 8th grade. I heard the high school children talking about algebra. I went to one of them, a tenth grade girl, and asked her what algebra was. She said, "Algebra is where you use letters." That left me more mystified than before and she went on to explain that the letters were x's and a^2's and b^2's, and that you multiplied and divided them. That was too much for me so I dropped the subject.

The following year I studied algebra. I have two clear cut recollections about it. In one I am standing at the board looking stupidly at a complicated fraction. (It surely "had letters in it.") The teacher comes along and asks, "What does that tell you to do?" I finally answer, "Divide." "Then why don't you do it?" In the other recollection, I happened to meet the good preacher of the village as I was on the way home with my report card in my hand. He looked it over, and said, "The algebra isn't very good." It was C. I wanted to tell him that algebra wasn't very interesting or important, and that it surely

did not connect with anything in which I was interested; but I didn't know how to say it, or quite dare say it to the Dominie.

Now my daughter has reached the time when the same question confronts her and her parents. What are you going to accomplish for her in the algebra class? If you say that you will prepare her so that after another half year's work she may hope to pass a college entrance examination, you do not arouse my enthusiasm very much. If she is the average ninth grade child, she is about one-tenth going to college and nine-tenths not going; and anyway that does not seem to me much of a promise for you to make us.

Possibly you are more modern and will tell me that you will give her the skill to perform in one minute, ten additions, or eight subtractions, or three multiplications, or four factorizations. You still fail to impress us very much. What should be the outcomes of ninth grade algebra teaching and what are the tests which should be applied?

I believe in the teaching profession and I believe in mathematics. Therefore I believe that answers to these important questions are very soon to be forthcoming. I believe that teaching is the greatest game in the world. I believe that teachers have the greatest influences in molding the life of a people. I believe in the value of straight thinking. I believe that knowing how to take hold of the quantitative side of life and to enjoy it; knowing how to attack its situations with confidence and success, to state its laws succinctly, and to talk about it with precision, are attainments of great value in such a civilization as ours. And because I have such faith in the teachers and in the mathematics, I believe that we are going to find the solution of our problem.

While we are waiting for the psychologists, experimentors, thinkers, and teachers to tell us what ninth grade outcomes and tests should be, perhaps we can see far enough ahead ourselves to sketch in some of them the bare outlines of the answer.

Suppose we say first that the best test of the teaching is in the attitude of the pupils. That the results of algebra teaching can be judged by *their* attitude toward the quantitative side of life. Do they expect to understand the "why" and the "what for" or only the "how"? Do they have the habit of expecting to understand.

You know the story of Mary, who reported to her father night after night that she was getting on all right in high school except in algebra and she didn't understand that. Some time later she dropped this formula and merely reported that everything was all right. "You understand your algebra now, don't you?" "Oh, no, father, but you don't have to understand it. You only do it the way they show you."

Is there any real education in doing algebra "the way they show you"? Is algebraic computation sufficiently valuable to justify our inflicting it upon all the children as a skill to be mastered in that way? No. It is the understanding which is educative; it is the understanding which is good for their minds. The understanding is more important than glibness: it is more important than the statement and use of rules: it is more important than facility in algebraic manipulation. To test algebra teaching we must test the interest, the understanding, and the attitude of the pupils. We are getting results in our algebra classes if the pupils like to come and if they are led to think, and if they are consciously building up an idea of what algebra is for and how the human mind has learned to take hold of quantitative relations. If the results are less than this, algebra teaching is not fulfilling its purpose.

II.

Our chief question is how to make algebra truly educative; how to make it accomplish this sort of thing for the minds of the young people. The present real progress in algebra teaching centers around the answer to this question. It is a matter of presenting material which is adapted to the child mind, and of presenting it in such a way that the child knows what it is for, and knows why he proceeds as he does; and so that he never has to hide behind mere rules for operation. Algebra is for him a subject which he expects to understand as he goes ahead step by step, and never the kind of thing which he does in the "way they show you." Algebra appears as a simple and direct language, very useful in stating facts and rules, and in asking and answering questions, and above all as a science in which everything proceeds according to the laws of common sense.

If this seems to you trite, will you not examine the algebra texts of yesterday and see if they do these things: see if they so

define algebra in the mind of the pupils. Far from it. In fact if we are to accomplish this real improvement in algebra teaching we must thoroughly revise the old order of presentation and the old method of presentation. That is our problem.

It is not a thing to be done lightly. Indeed it has been occupying the attention of "our best minds" for a period of more than twenty years. There is time this morning to touch upon only a few of the important conclusions arrived at.

III.

With what should ninth grade algebra begin?

The new algebra begins with the question, What is algebra for? and it discovers little by little the answer to the question. The use of each process becomes evident at least as soon as the process itself. In fact the order usually is this; first a new mathematical difficulty; second a study of the situation so that we may find the way out; third the discovery of the new process which overcomes the difficulty. This of course throws into the foreground the formulas and equations and the simple manipulation necessary to compute with them. The four processes which used to fill so many of the opening pages of the text are now removed far to the rear, and in their complex cases, omitted entirely. It was of course the worst possible psychology to attempt to educate the child by requiring him to manipulate symbols which had for him no significance.

IV.

Equations should be transformed without juggling or artificial devices of any kind, multiplications, subtractions, additions, and divisions should be used to simplify them step by step according to the laws of common sense. This is the only way in which we can keep the pupil in the understanding frame of mind.

There has been much discussion of the question when negative number should be introduced. The answer is that it should not be introduced as a topic anywhere. Instead the meaning of the minus sign should be allowed to grow naturally, step by step, as occasion arises. The pupil will readily enough take each step at the moment when he needs it. The treatment of these three ideas: what equations are for, how equations are handled, and

the idea of negative number, is fundamental to the present improvements in algebra teaching, and these ideas are so interrelated that we can best consider them together.

Formulas state facts briefly. They give us a contact with arithmetic and with science, and lead us into many interesting fields. Of course we should not foist upon the pupil a hodge podge of formulas. There is little value in that. Rather we should investigate thoroughly a few formulas which he can really understand and use.

Equations ask and answer questions. They begin with the form $5x = 20$. If we know the value of five x's we must divide in order to find the value of one x. It is curious that we thought that anything was to be gained by bringing the strange word *axiom*.

In the form $2x + 5x = 35$ we must unite. Say to the pupil, "Two x's and five x's give ——." This completely does away with the question, What becomes of the other x when we unite $2x$ and $5x$ and get $7x$. It fixes the attention on the meaning of the symbols.

In the equation $3x + 8 = 20$ we know what $3x$ and 8 equals. We want to know what $3x$ equals, consequently we must *subtract* 8. It is strange that we should have thought it necessary to bring in the balances to over-illustrate anything so simple. The child knows that if two numbers are equal and he takes 8 from each number he will get equal results. If we leave him alone merely cautioning him to treat both sides alike, he will come out all right.

It is very curious that in the old days persons who considered themselves teachers should have introduced transpositions into elementary algebra. It is an enemy of thought just at the moment when what we must want is thought. If we teach the child to transpose we drive him at once from the understanding attitude of mind over to the juggling attitude where he must hide himself behind a rule or a trick. Either algebra is for him a subject in which he reasons his way along by common sense or it is a subject in which he follows rules more or less blindly. If we let him reach this second point of view we defeat the major aim of algebra teaching.

Some text book writers, either because they failed to think the problem clear through, or because they feared that teachers were

not ready to follow them, have tried to compromise by explaining once that equations can be simplified by addition and subtraction and then introducing transposition as a short cut. This is of course a failure to recognize the difference in the psychology of teaching for an understanding and teaching for mechanical mastery. The chief end of ninth grade algebra teaching is understanding and the only way to teach an understanding is to understand over and over again, and to think out the process until right reaction becomes habitual.

An eminent psychologist says that of course we do not want the pupil laboriously to think of the additions and subtractions in such cases. But even eminent psychologists can err, as they themselves would be the first to admit. In the first place thinking of additions and subtractions is not noticeably more laborious than thinking of numbers that fly up plus and come down minus. In the second place if by laborious the professor means thought compelling, that is exactly what we do want, and what we must have if we are to make algebra educative.

In the case $5x - 8 = 48$ we have a minus sign but it means about what it always has meant. It means to subtract the 8. Since we cannot actually perform the subtraction, we indicate it. Perhaps the best name for such a suspended subtraction is a shortage. We do not have the whole of $5x$; we are 8 short of $5x$. We want to know what $5x$ equals so we add enough to make up the shortage. This idea of a shortage is the only extension of the meaning of the minus sign which is needed for a long time.

In $5x - 8 - 6 = 17$ we want to combine two shortages. Once the idea is brought to the pupil's attention he will do it correctly. In uniting positive and negative numbers he carries out the same idea, needing only very simple illustrations of profits and losses, or gains and losses in a football game. One great difficulty, when we thought that we had to teach negative number as a topic, was that we over-illustrated and over-explained, thereby confusing the child. We made a mountain of a molehill and insisted that he climb over it.

The pupil is interested to discover that he can now subtract a large number from a smaller one, as he could not do in arithmetic. $5 - 8$ leaves him *short* 3. That is easy. That is all there is to it.

a shortage is really adding. $5 - (-3)$ means $5 + 3$. The idea of a debt or a shortage leaves the pupil in no doubt about the matter. For subtraction we may then give the mechanical rule *provided* that we frequently require the pupil to explain why he operates as he does.

Our guiding principle is that merely teaching a child to carry out a rule for algebraic manipulation is not educative. The practical value of the rule is not sufficient to justify it. But teaching the child mind to understand, to want to understand, to expect to understand, is education.

Soon we need the parenthesis, as when we want to multiply in the expression $5\ (x - 3)$. (Parenthesis are of course no longer taught as a topic but are *used* when they are needed.) $5\ (x - 3)$ means that we must multiply both x and -3 by 5. Fives times a shortage of 3 leaves us *short* 15.

The expression *remove the parenthesis* should no longer persist. What we want the child to do is not to remove anything but; 1. To understand the directions. 2. To carry out the directions.

In $5\ (x - 3)$ he is told to multiply. He considers how to do it, then he does it. There is no idea of removing anything. In $4 + (x - 3)$ he adds. In $6 - (2x + 5)$ he subtracts. In $\frac{3x - 9}{3}$, in which the bar of the fraction serves as his parenthesis, he divides. In $(x + 3)^2$ he raises to a power. In $\sqrt{a^2 + 2ab + b^2}$, in which the vinculum serves as his symbol of aggregation, he extracts a root.

It is a serious mistake to associate the parenthesis with any one process or to talk about the operation of removing it. The parenthesis merely ties together two or more terms; we must look elsewhere to see what we are told to do to these terms. It may be add, subtract, multiply, divide, or take a root, or raise to a power.

Thorndyke says that numerous experimenters besides himself have found that making the general topic of removing parentheses subordinate to multiplication rather than to addition and subtraction decreases the difficulty. One hesitates to take issue with so able and so serious a student of the teaching of algebra

but it is strange that Professor Thorndyke should have formed such a series of bonds as is indicated by this reaction. In the first place we do not want to remove parenthesis, but to operate on the terms therein as we are directed to do. One wonders how to apply the above dictum to $(a + b)\ (a + b^2)$. Doubtless he is thinking only of such cases as $23 - (4x + 3)$ and $15 + (2x - 6)$ in which he would have us multiply in the first case by -1 and in the second, I suppose by $+1$.

Now let us see if we cannot undermine his position in this case also. If he claims that the aim in algebra teaching is the quickest way to get the child to operate, he may have some ground to stand on, as his experiments may show. (How the Quickest Way Delusion has spoiled good teaching in the high school.) But if we believe that we should teach the child that algebra is a beautiful language which states facts and asks and answers questions, and gives beautifully definite directions about what operations to perform, and that it operates always by the laws of common sense; why then the Professor does not have a leg to stand on. $23 - (4x + 3)$ says to the pupil to subtract, and $15 + (2x - 6)$ says to him to add.

And I am not through with Prof. Thorndyke yet. In $-4\ (x - 5)$ he wants the child to think of $-4\ (-5)$, that is of the product of two negative numbers. That is an operation which cannot take on any meaning for the child. He cannot think of any such operation. He must protect imself behind a rule. (Of course the operation can be illustrated as in motion east and west in time past or present, but any such explanation is too elaborate to be a carried around in the pupil's mind when he is performing the operation.) The way in which the child should think about the matter is this: "In $5x - (x + 3)$ I am told to take away $x + 3$. To do this I must take away both x and 3, writing $5x - x - 3$. In $5x - (x - 4)$ I am told to subtract x less 4. When I write $5x - x$ I have subtracted too much. I have subtracted 4 more than I was told to subtract, hence I must put back 4 and write $5x - x + 4$." In this way the child is still traveling on his own feet, guided by his own common sense.

In $x - 4(x - 3)$ he says, "I am to do two things. I am to multiply $x - 3$ by 4 and then I am to subtract the result. Multiplying I get $x - (4x - 12)$. Subtracting I get $x - 4x + 12$.

After he has become familiar with such multiplications and with their explanations (but not too soon) there is no harm in showing him how he can multiply and subtract at one step. Such a process is one that he can understand. He can have a rule for taking these two steps at once when the signs are both negative; for the other cases he needs no rule, common sense is better. Whatever rule he does use he should be able to justify whenever called upon to do so. If the multiplication of two negatives is to be a thing understood it must be thought of as a multiplication and a subtraction.

Somewhere along here the pupil will be confronted with the question, can a problem have a negative answer? Perhaps the problem will deal with a temperature below zero or with profit and loss. In any case this problem will be sufficient, when it is associated with graphs in other quadrants than the first, to show that the minus sign sometimes means, "in the opposite direction." This completes the pupil's extension of the meaning of the minus sign. He will tell you that it has three meanings. It indicates subtraction. It indicates a shortage. It means "in the opposite direction." With these three conceptions he can explain any of his operations.

The secret of success in teaching negative numbers is in letting these meanings develop slowly in the mind of the pupil, letting each new idea come up when it is most helpful, and insisting on an understanding of each process. During the same period there will be growing also the conception of what equations are for and how one understands them and puts them to work.

The new algebra is very different from the old because teaching for understanding is very different from teaching for me. chanical mastery only; and because the new psychology has taught us so much about how the minds of young people may be educated.

We no longer consider it right to begin the year with the manipulations of symbols which have no meaning. Symbols are not very interesting things in themselves, when you operate with them without knowing what they mean or what they are for, and particularly when you operate by means of ready made rules which you use because you are told to do so.

There is no time to talk about the new symbolism which impels the pupil to think first of what he is to do and to set down

the actual operation he is to perform before he starts to do it. In ① $3x - 8 = 4$ the pupil writes the directions ① $+ 8$, and then proceeds to add 8 to both sides of his equation.

In ① $x + 4y = 9$ he writes ① — ② and then proceeds to subtract.
② $x - y = 3$

In ① $\frac{3}{x} = \frac{8}{4}$ He writes (1) $\times 4x$ and then proceeds to multiply.

I should like to speak further of the solution of linear equations in one unknown, by a series of logical steps, multiplications, additions, subtractions, and divisions with no hint of juggling. This can be done by what is known as the rule of Dioptantus, and it must be done if we are to give to algebra that real educational value which will confound the critics of high school mathmatics.

I should like to speak of the duty of the ninth grade teachers to improve the abilities of her pupils in numerical computation. How she should bring into play their growing judgment, and leave them at the end of the year with considerable ability to "figure" as practical computers do. For this I shall have to refer you to the little book Teaching Junior High School Mathematics, Houghton Mifflin, Boston.

If there was time we could talk over the dangers of going too far with new generalizations, at the moment when we are getting rid of the old generalizations which have been found to be unfitted for ninth grade understanding. What I refer to is such a thing as the over-emphasis of the generalization which we call the function concept.

These matters and many others will pass in review before those of you who are studying the problems of the new, more interesting and more valuable algebra.

VI.

Those of you who are largely concerned with college preparation have a right to ask if the new algebra prepares for college. In order to answer this question an extensive experiment was made, in which more than two thousand pupils, so chosen as to have a very wide range of I. Q's., were taken through their high school mathematics in the way we have been considering. More

than eleven hundred of them went on to higher institutions, the larger part of them to three very well known universities. The university authorities knew nothing of the experiment as such.

The two institutions receiving the largest numbers of these pupils reported on their success without being asked to do so. One said, "We should like to know what you do to these pupils to make them stand up to our work better than any other pupils we get." The second said, "How can we get more pupils so prepared?"

The third institution was asked for a report on the success of these pupils. They replied that an investigation showed these pupils to possess "unusual mathematical competence."

VII

The real improvement in mathematics teaching which is occurring at the present time will be characterized perhaps in future histories of education, as having its origin in the new understanding of the learning process in young people: and as completely rearranging the subject matter and method of presentation so as to aim throughout at the development of understanding and of mental power.

ANNOUNCEMENT

An Investigation in the Teaching of the Skills of Ninth Grade Algebra.

By Raleigh Schorling and Selma A. Lindell

University High School, Ann Arbor, Michigan

Purpose—The purpose of this study is to measure the effect of applying certain principles from the psychology of drill. In recent years considerable gain has been achieved from our knowledge of habit formation. It is now possible to formulate over twenty principles that are generally accepted in psychology but which are not applied in our school texts in algebra.

These principles may be illustrated by the following:

(1) There should be much practice on a few skills rather than a little practice on each of many things.

(2) A drill exercise must be specific.

(3) Drill, to be effective, must be individual.

(4) Drill must provide a scoring technique so that the pupil may know the degree to which he succeeds.

(5) Drill should make possible the diagnosis of individual difficulties.

(6) Practice should be distributed in decreasing amounts at increasing intervals.

(7) A drill exercise should be standardized so that the pupil may have at least a rough notion of his degree of skill.

(8) A drill exercise should be relatively short. Probably an approximate time for concentrated work in the ninth grade is between eight and ten minutes.

(9) A drill exercise should set up a high standard of accuracy. The scheme for drill should make it possible for the pupil to take the same exercise again and again until he has achieved an accuracy that allows only the occasional slip.

The foregoing is probably a list sufficient to illustrate the very definite guiding principles for habit formation that the investigators have in mind. How effective are these principles in contrast with the usual "hit or miss" plan.

Experimental Material

The experimentors, with the help of Dr. John R. Clark, have constructed a booklet of teaching materials which aims to apply

consistently and throughout these well recognized principles of drill. This booklet is entitled "Instructional Tests in Algebra." It consists of fifty-two goals and is adjusted for pupils of varying abilities.

General Plan of this Experiment

(1) A teacher taking part in this experiment will teach two or more algebra classes. During the first month all classes will be taught in the usual way the material in some standard algebra text already in use in the particular school system concerned.

(2) Beginning with the fifth week the Experimental classes will take ten minutes each day at the beginning of the hour to use the experimental booklet described above. The rest of the recitation period will be spent on conventional materials drawn from the regularly used text books.

(3) It is important to note that each teacher taking part in this experiment will have at least one *Control group*, that is, she will teach one class without modifying in any way the course that is regularly given in that city.

(4) An algebra test will need to be given to all groups at the beginning of the year. Considering the fact that all pupils will be beginning algebra it might seem that this step is unnecessary. There are at least two reasons why a test is essential. In the first place there will be repeaters. In making the final report at the end of the school year all papers of pupils repeating algebra either in the experimental class or the control class should be rejected. In the second place, many cities are now studying general mathematics in grades 7 and 8. In these courses pupils learn some algebra, and it is conceivable that bright students might bring considerable skill in certain aspects of algebra to the 9th year course.

(5) Comprehensive tests will be given at the end of the school year to both the Experimental and Control classes. This test will consist of parts drawn from the various standardized tests in algebra.

How Materials May Be Obtained

(1) The test for the beginning of the school year consists of two parts: the Addition and Subtraction Test and the Equation and Formula Test by Hotz. These tests may be secured from Dr. Clifford Woody (Ann Arbor, Mich.) and charged to Dr. Schorling. Please mention this experiment in asking for these tests.

(2) The *Experimental Booklet* can be obtained by writing to George Wahr, Publisher, Ann Arbor, Mich. The 72 page booklet is furnished at a cost of 30 cents per pupil.

(3) At the end of the year a co-operating school may administer to both groups the complete form of any one of the following tests: 1. The Hotz First Year Algebra Scales, (Teachers College Bureau of Publications, New York City). 2. Diagnostic Tests for First Year Algebra by Professor Harl Roy Douglass. (For the test write Professor Douglass, University of Oregon, Eugene, Ore.) 3. The Thorndike Algebra Tests—To Quadratics Form A. (For the test write the Institute of Educational Research, Teachers College, New York City).

In some schools it may be difficult to administer a formal test such as is mentioned in the preceding paragraph. For such schools a *special final* test will be supplied.

This final test is to be specially constructed for measuring the outcome of one year of Algebra in the two groups concerned.

The test will be furnished on May 15th without cost by the investigators upon special request of a co-operating teacher. For this test write the experimentors direct.

Conditions of this Experiment

This investigation will be conducted intensively in a limited number of schools in which it is arranged:

(1) To give the tests at the beginning of the year to both groups.

(2) To administer the Experimental Booklet according to directions provided in the booklet.

(3) To administer a comprehensive test at the end of the school year to both groups.

(4) To write to the investigators before May 15, 1926, for the blank form on which the results of the experiment will be reported to the experimentors.

Those who are especially interested in this experiment should notify the investigators and make certain that they have the Experimental Booklet at the beginning of the fifth week of school this fall.

Signed,

RALEIGH SCHORLING
SELMA A. LINDELL

THE FIRST YEARBOOK OF THE NATIONAL COUNCIL OF MATHEMATICS TEACHERS

At the last annual meeting at Cincinnati the Council voted to publish a yearbook containing helpful materials for teachers of Mathematics. This yearbook is to be presented as the program for the 1926 meeting. In accordance with this action the president of the National Council appointed the following committee: Mr. Charles M. Austin, Chairman, Dean John H. Minnick, Mr. William Betz, Professor Walter E. Eells, and Professor Frank Touton.

The next meeting will be held at Washington, D. C., on Saturday, February 27th. It is hoped at this time to present a strong program and an excellent yearbook.

The details concerning content, publication, and distribution will need to be approved by the Executive Committee of the Council. It is proposed that we build a yearbook around the general theme, "The Progress of the Past Twenty-five Years in the Teaching of Mathematics in Elementary and Secondary Schools." Credit for this suggestion is due Mr. William Betz.

Tentative plans for the yearbook include the following:

1. The Presidential Address by Professor E. H. Moore. This is a reprint of the address of Professor E. H. Moore delivered before the American Mathematical Society. It has effectively stimulated progressive thinking on the teaching of High School Mathematics.

2. Twenty-five Years of Progress in the Schools of England, a summary by Professor T. Percy Nunn.

3. Recent Tendencies in the Schools of Germany by Professor J. W. Young.

4. A summary consisting of, (1) the story of our syllabuses, (2) the work of the International Commission, (3) the story of the National Committee and its work, (4) the attempts at curriculum construction, etc. It is suggested that this summary be undertaken by one of the following: Professor J. W. A. Young, Professor D. E. Smith, Professor G. W. Myers, or Mr. William Betz.

5. Measuring Instruments of Long Ago, a reprint of Mr. Stark's article.

6. A summary of the present testing movement with a *critique* of tests. It is hoped that Professor Clifford B. Upton will undertake this as he needs only to abbreviate and supplement his material of Chapter XIII in the *Reorganization of Mathematics in Secondary Education.*

7. The Making of a Modern Survey by Professor Raleigh Schorling.

8. A statement of the Modern Point of View as to the Psychology of Drill by Raleigh Schorling.

9. A Biography of books published since 1920 by Professor J. Robert Overman.

10. The Teaching of Trigonometry in the Junior High Schools by Mr. Richard McDaid.

11. By Products from the movement in education as concerns the teaching of Mathematics. It is hoped that one of the following may undertake this unit: Professor E. L. Thorndike, Dr. John R. Clark, or Dr. Clifford B. Woody.

12. The Teaching of the Metric System in Relation to the Teaching of Decimal Fractions, Miss Selma Lindell.

13. What are we able to do in Mathemtatics by way of diagnosis and remedial treatment? Leroy Schnell.

No one man or a publication committee can make a yearbook. We will need the judgment of many in our field. You are hereby urged to criticize the preceding outline and to make suggestions as to the other units to be included. Communications should be forwarded to Professor Raleigh Schorling, President of the National Council of Teachers of Mathematics, University of Michigan, Ann Arbor, Michigan, or to Mr. Charles M. Austin, Chairman of Yearbook Committee, High School, Oak Park, Ill.

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Teachers Library, Senior High School, Fourth and Limestone Streets, Lexington
Prof. A. C. Kuykendall, Transylvania College, Lexington
Library, University of Kentucky, Lexington
Jefferson Davis Junior High School, South Limestone, Lexington
J. M. Atherton High School for Girls, Morton Avenue at Rubel, Louisville
Central Colored High School, 9th and Chestnut, Louisville

Louisville Girls' High School, 506 W. Hill Street, Louisville
Louisville Male High School, 905 South Brook, Louisville
Emma A. Lang, 1061 Cherokee Road, Louisville
Mary S. Barret, 404 W 9th Street, Owensboro
C. R. Clark, Pikesville High School, Pikesville
Nellie M. Hubert, 523 S 8th Street, Paducah
Delia Covington, Russellville
J. T. Vallandingham, Williamsburg

LOUISIANA

Bolton High School, Alexandria
W. Paul Webber, 211 Poland Street, Baton Rouge
State Department of Education, P. O. Box 633, Baton Rouge
Hill Memorial Library, La. State University, Baton Rouge
Superintendent R. V. Kerr, Benton
Mary Schremp, Bonnabel Pl., Metairie Ridge, Jefferson Parish
Ila Proffer, Jennings
Miss Dora M. Forno, 7924 Jeanette Street, New Orleans
Newcomb College Library, Dixon Hall, Oak & Broadway, New Orleans
George Longe, 3500 Hamburg Street, New Orleans
Isadore Newman, Manual Training School, New Orleans
Winnie D. Daly, S. B. Wright High School, Napoleon Avenue and Prytania Street, New Orleans
Caroline S. Wischan, S. B. Wright High School, Napoleon Avenue and Prytania Street, New Orleans
Irene G. Hagen, 117 N. Genois Street, New Orleans
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MAINE

Arloene Farnham, 259 Court Street, Auburn
Margaret B. Jordan, 91 Pleasant Street, Auburn
Bernice B. Dunning, 156 Cedar Street, Bangor
Harriet Wentworth, 22 Summer Street, Bangor
Milon J. Herrick, 97 Main Street, Bridgton
Errol Dearborn, care Farmington Normal School, Farmington
Harold B. Swicker, 24 Oak Street, Guilford
Bates College Library, Lewiston

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